Parallel Graded Generalized Delaunay Mesh Refinement*

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Abstract

We describe a complete solution for both sequential and parallel construction of guaranteed quality Delaunay meshes for general two-dimensional geometries. We generalize the existing single-core point placement strategies for guaranteed quality mesh refinement: instead of a specific position for a new point, we derive a selection disk inside the circumdisk of a poor quality triangle. We prove that any point placement algorithm which inserts a point inside the selection disk of a poor quality triangle will terminate and produce a guaranteed quality size-optimal mesh. In the area of parallel Delaunay mesh refinement, we develop a new theoretical framework for the construction of graded meshes on multi-core architectures, i.e., meshes with element size controlled by a user-defined criterion. Our sufficient conditions of point Delaunay-independence allow to select points for concurrent insertion in such a way that the standard sequential guaranteed quality Delaunay refinement procedures can be applied in parallel to attain the required element quality constraints. Finally, we present a novel multi-core algorithm which can be used in conjunction with any sequential point placement strategy that chooses points within the selection disks. We implemented our algorithm for shared memory multi-core architectures and present the experimental results. Our data show that even on workstations with a few cores, which are now in common use, our implementation is significantly faster than the best sequential counterpart.

Delaunay refinement is a popular technique for generating triangular and tetrahedral meshes for use in the finite element method, the finite volume method, and interpolation in various numeric computing areas. Among the reasons of its popularity is the amenability of the method to rigorous mathematical analysis, which allows to derive guarantees on the quality of the elements in terms of circumradius-to-shortest edge ratio, the gradation of the mesh, and the termination of the algorithm. The parallelization of two-dimensional mesh generation algorithms is particularly important for some three-dimensional simulations which use multiple two-dimensional meshes in different coordinate systems. Some examples include direct numerical simulations of turbulence in cylinder flows with very high Reynolds numbers, see [3], and coastal ocean modeling for predicting storm surge and beach erosion in real-time, see [10]. For the modeling of turbulence, as shown by Karniadakis and Orszag [6], with the increase of the Reynolds number Re, the size of the mesh grows in the order of $Re^{9/4}$, which motivates the use of parallel mesh generation algorithms. At the same time, the size of the mesh should be as small as possible given the required element quality constraints, which can be attained by using a nonuniform (graded) mesh.

We address theoretical and practical aspects for the development of both single- and multi-core parallel Delaunay mesh generation algorithms and software that satisfy the following requirements:

1. allow to construct well-shaped elements with bounded minimal angle;
2. produce graded meshes, i.e., meshes with element size specified by a user-defined function;
3. offer proofs of termination and size optimality;
4. allow to use custom point placement strategies (e.g., circumcenter, off-center, etc.);
5. replace the solution of a difficult domain decomposition problem with an easier data distribution approach without relying on the speculative execution model;
6. offer performance improvement over the best available sequential software, even on workstations with just a few hardware cores.

The field of sequential guaranteed quality Delaunay refinement has been extensively studied, see [5] and the references therein. However, new ideas and improvements keep being introduced. One of the basic questions is where to insert additional (so-called Steiner) points into an existing mesh in order to improve the quality of the elements. Frey's [4], Ruppert's [7], and early Chew's [1] algorithms use circumcenters of poor quality triangles. Later, Chew [2] suggested to use randomized insertion of *near-circumcenter* points for three-dimensional Delaunay refinement, with the goal of avoiding slivers. Recently, Üngör [9] proposed to insert specially chosen points which he calls *off-centers*; this method allows to produce smaller meshes in practice and it was implemented in the popular sequential mesh generation software the Triangle [8]. We expect that other optimization techniques can be used to select positions for new points. We

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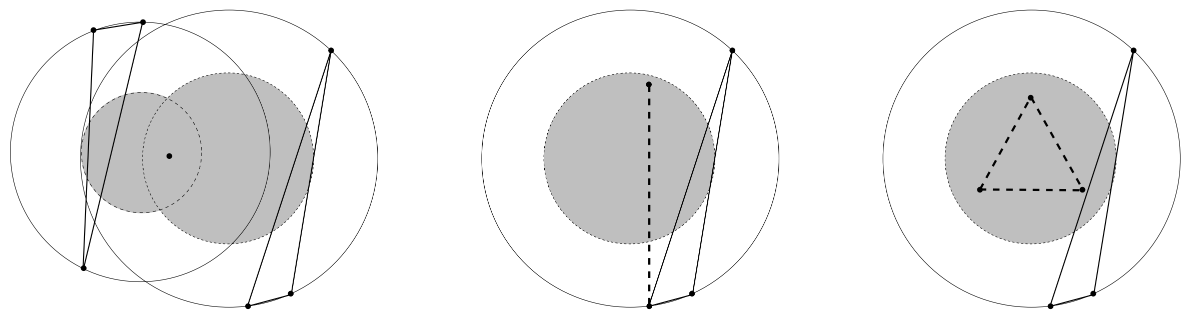

Figure 1: Examples of the approaches for choosing Steiner points within selection disks of skinny triangles.

give an example of a point placement strategy which in some cases allows to achieve even smaller meshes than the off-center method, albeit at significant computation cost. Since one would not like to redesign the parallel algorithm and software to accommodate each of the point placement techniques, in this paper we generalize the sequential Delaunay refinement approaches and develop a framework which allows to use custom point selection strategies. In particular, we derive a *selection disk* for the position of a new point with respect to a poor quality triangle and prove that any point placement technique with the only restriction that it selects a point inside the selection disk will terminate and produce a size-optimal guaranteed quality mesh. While the use of Chew's [2] *picking-sphere* is restricted to produce only meshes with constant density, the use of our selection disk allows to obtain graded size-optimal meshes.

We conducted experiments with three different point placement methods: circumcenter, off-center and a new optimization-based method which allows to improve the size of the mesh by up to 20% and up to 5% over the first two methods, respectively. We analyzed the existing point insertion methods for guaranteed quality Delaunay refinement and unified them into a framework which allows to develop customized mesh optimization techniques. The goals of these techniques may include the following:

1. eliminating slivers, see [2];
2. splitting multiple poor quality triangles simultaneously, e.g., to minimize the number of inserted points, see Fig. 1(left).
3. creating elongated edges in required directions, see Fig. 1(center);
4. inserting more than one point, e.g., to create elements with specific shapes, see Fig. 1(right).

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